SHOTTKY SIGNALS FOR LONGITUDINAL AND TRANSVERSE BUNCHED BEAM DIAGNOSTICS

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Abstract
Following a brief historical overview on the origin and the evolution of Schottky noise we discuss applications in the field of beam diagnostics in particle accelerators. A very important aspect of Schottky diagnostics is the fact that it is a non perturbing method. Essentially statistics based, it permits to extract beam relevant information from rms (root mean square) noise related to the movement of the individual particles. This is also the basis for stochastic cooling. Schottky diagnostics permits one to extract a considerable number of important beam parameters such as the revolution frequency, momentum spread, incoherent tune, chromaticity and emittance.

INTRODUCTION
In the year 1918 the German physicist Walter Schottky (* 23 July 1886 in Zurich) published a paper describing the mechanism of spontaneous current fluctuations in different conductors. This is the origin of the term Schottky noise. The term Schottky noise refers both to thermal noise in resistors and noise in charged particle beams. Additional important contributions from Walter Schottky are the Schottky diode, Schottky defects (in semiconductors) and the Schottky equation (Langmuir – Schottky equation for space charge). In 1915 Schottky invented the tetrode and in 1918 he pioneered the superhet concept. In 1928 the thermal noise in resistors was first measured by J.B. Johnson (Bell Labs) and he discussed his findings with Harry Nyquist who worked at the same laboratory. This is the origin of the term Johnson-Nyquist noise which is more frequently used in the English literature when referring to thermal noise.

An important milestone in accelerator technology was the invention of the stochastic cooling concept in 1968 by Simon van der Meer (Nobel price shared with Carlo Rubbia in 1984). Clear Schottky noise signals from a strong coasting (unbunched) beam of protons were observed in 1972 in the CERN–ISR (Intersecting Storage Rings) followed in the same year by the first publication of the cooling idea by Simon van der Meer [1]. In 1975 schemes for pbar accumulation were developed and tested experimentally with protons in 1976 (ICE = Initial Cooling Experiment). Over the following years a rapid worldwide evolution of beam diagnostics with Schottky noise took place, both for bunched and unbunched beams. Nowadays Schottky diagnostics is a vital element in nearly all large circular machines operating with hadrons and also to a certain extent for electron rings and even linacs. The information extracted this way allow continuous monitoring of important beam parameters and the control of a number of related machine settings.

SHOT NOISE IN A VACUUM DIODE
Consider a simple vacuum diode (fig 1) where a small number of electrons pass from a heated cathode to the anode [2].

![Vacuum Diode Diagram](image)

Fig. 1 a: Vacuum diode with two electrodes (from [2])
Fig. 1 b: Anode current related to individual electrons

When a single electron is emitted from the cathode and starts moving to the anode (due to the acceleration voltage $U_0$) an approximately linear increase of current at the anode is measured. This is due to the $dD/dt$ ($D$ = dielectric displacement) related displacement current which continues as a conducting current when the electron approaches the flat anode. Each of these saw tooth like signals (fig 1b) has a length $\tau$ which is the travel time of the electron from the cathode to the anode. As the individual electrons are emitted in a random manner, those saw tooth like signals occur as a non periodic time function. This is very similar to the time function of acoustic noise originating from little grains falling on a metal plate and is the origin of the term “shot noise”.

If we assume this diode to be working in the saturated regime (i.e. the anode voltage is high enough that all emitted electrons are accelerated to the anode and there is no space charge cloud near the cathode) then, after some
derivation, the “low frequency spectral density $S_f(\omega)$ of the short circuit current can be expressed as

$$S_f(\omega) = 2I_0e$$

(1)

where $e$ is the elementary charge of an electron with $v_{mean} = N/\tau$ and $I_0$ stands for the mean current as defined by $I_0 = e v_{mean}$.

Obviously the travel time $\tau$ plays a very important role for the range of validity of this formula. Typical $\tau$ values for vacuum diodes operated at a few 100 volts and a cathode anode spacing of around a cm are in the order of a fraction of a nanosecond. This translated to maximum frequencies in the GHz range (fig 2).

In a similar manner, starting from the saturated high vacuum diode, via the vacuum diode in the space charge region and then applying the theory [2] to a biased and unbiased solid state diode one can finally arrive at the noise properties of a linear resistor. This general relation for the open (unloaded) terminal voltage $u$ of some resistor in thermo dynamical equilibrium for a frequency interval $\Delta f$, which is also valid for very high frequencies $f$ and/or very low temperatures $T$ (K), is given by

$$u^2 = 4k_BTR \frac{hf/k_BT}{\exp(hf/k_BT) - 1} \Delta f$$

(2)

Here $h=6.62 \times 10^{-34}$ J·s is Planck’s constant and $k_B = 1.38 \times 10^{-23}$ J/K equals Boltzmann’s constant. From this general relation it is possible to write the low frequency approximation, which is still reasonably valid at ambient temperature up to 500 GHz, as (open terminal voltage)

$$u^2 = 4k_BT \Delta f / R$$

(3)

and for the short circuit current we get accordingly

$$i^2 = 4k_BT \Delta f / R$$

(4)

Note that the factor 4 in equations (3) and (4) often leads to confusion. When terminating a noisy resistor with an “external load” consisting of a resistor of the same ohmic value (power match) but at 0K temperature one can visualize the “available noise power”. This power delivered to an external load is given by

$$P = k_BT \Delta f$$

(5)

and is independent of the value of $R$! For the power density $p$ per unit bandwidth an even simpler relation is obtained

$$p = k_BT = -174 dBm/Hz \ @300K$$

(6)

This relation is also valid for networks of linear resistors with a homogeneous temperature between any two terminals. In fact a normal carbon resistor is already such a network of many tiny carbons grains. Equation (6) does not apply to resistors or resistor networks which are not at a homogeneous temperature. In particular, devices which are not in thermo dynamical equilibrium such as a DC biased diode or a transistor connected to some supply voltage are excluded. Such “active” resistors may have noise temperatures which are considerably below their physical temperature and can be used as a pseudo-cold loads in order to avoid bulky and costly cryogenics [3]. As an example one may take the typical TV satellite receiver front-end where the actual input amplifier has a noise temperature of about 50-70 K in the 10-12 GHz range. In the CERN LEIR machine the concept of pseudo-cold terminations in the 100 MHz range has been applied for strip-line type pick-ups [4].

**THE FIELD SLICE OF FAST AND SLOW BEAMS**

Similar to what has been shown for the planar vacuum diode, where the spectrum is a function of the electron velocity and the spacing between cathode and anode, particles in a particle accelerator produce $\beta=\frac{v}{c}$ related spectral modifications in the beam-pipe. This is linked to the fact that virtually all Schottky monitors interact in one way or the other with the image current on the inner surface of the beam-pipe. For slow beams the “field slice” has a certain opening angle (roughly $\sim 1/\gamma$) which causes a “smear out” of the spatial resolution and thus limits the maximum observable frequency as a function of beam-pipe diameter and $\gamma$-value. Figure 3 gives a nice impression how the “field slice” contraction for increasing beam velocity enhances the spectral content in the image current towards higher frequencies.
SINGLE PARTICLE CURRENT

Consider a single particle circulating in some storage ring with a constant frequency $\omega_0 = 2\pi f_0 = 2\pi/T$. This particle will induce a certain signal on some pick-up at its passage time $t_k$

$$i_k(t) = \frac{e}{T} \sum_{m} \delta(t - t_k - mT)$$  \hspace{1cm} (7)

Applying the Fourier expansion to $i_k(t)$ gives

$$i_k(t) = i_0 + 2i_0 \sum_{n=1}^{\infty} a_n \cdot \cos n\omega_0 t + b_n \cdot \sin n\omega_0 t$$

with $i_0 = e\omega_0$, $a_n = \cos n\varphi_k$ and $b_n = \sin n\varphi_k$

This leads to a corresponding series of Dirac pulses in the frequency domain.

With a second particle at a slightly different frequency $f_1 = f_0 + \Delta f$ with

$$\Delta f = f_0 \times \eta \frac{\Delta p}{p}$$ \hspace{1cm} (9)

one obtains the situation shown in Fig 5. The frequency difference $\Delta f$ at each harmonic of the revolution

LARGE NUMBER OF PARTICLES IN LONGITUDINAL PHASE SPACE

Now we have to deal with the problem that we have a large number $N$ of particles with (for the moment) equal revolution frequency, but random initial phase $\varphi_k = \omega_0 t_k$. This will result in some amount of mean current $I_0 = N i_0$, which is proportional to the number of circulating particles in the machine and which does obviously not depend on the initial phase of each individual particle, and fluctuations caused by the random phases

$$I(t) = I_0 + \Delta I$$

where the fluctuations are given by

$$\Delta I = \sum_{n=1}^{\infty} I_n = 2i_0 \sum_{n=1}^{\infty} A_n \cdot \cos n\omega_0 t + B_n \cdot \sin n\omega_0 t$$

$$A_n = \sum_{k} \cos(n\varphi_k) \hspace{1cm} B_n = \sum_{k} \cos(n\varphi_k)$$  \hspace{1cm} (10)

In the total current $I(t)$ the n-th harmonic contains the contributions of all $N$ particles. On a time display one can only recognize a noise trace with some DC offset. But in the frequency domain distinct bands (Fig 5) occupied with noise would become visible. Notice that for the current fluctuations the amplitude $\Delta I$ shown in Fig 6 is a $\pm 1\sigma$ value and that the peak excursions seen on a scope will be much higher due to the fact that we have a Gaussian amplitude density distribution [6].
Omitting a few intermediate steps in the derivation, one obtains the following expression for the mean square current fluctuations contained in the n-th harmonic

$$\langle I_n \rangle^2 = \frac{(2ie_0)^2}{2} \sum_{k=1}^{N} \cos^2 n\varphi_k + \sin^2 n\varphi_k = 2e^2 f_0^2 N$$

or

$$\langle I_n \rangle^2 = 2e^2 f_0^2 N = 2eI_0 f_0 \quad [A^2] \quad (11)$$

In this relation, which describes the probable power contribution to the n-th Schottky band for a group of \( N \) mono-energetic particles, results in a set of equal lines in the frequency domain, spaced by \( f_0 \). There is no dependency on the harmonic number. The current fluctuations (Fig 6) now appear as:

$$I_n = \text{constant} \times \cos(\omega_n t + \varphi_n)$$

with \( I_{rms} = 2e f_0 \sqrt{N} \quad (12)$$

In the case of ions with charge number \( Z \) (i.e. each ion presenting a “macro-particle” with charge \( Z e \)), the \( e \) in the above equation is replaced by \( Z e \):

$$\langle I_n \rangle^2 = 2(Ze)^2 f_0^2 N$$

(13)

Note that the power in the fluctuations is proportional \( Z^2 \) which means that a single fully stripped uranium ion gives the same signal as about 8500 protons.

Of course it is not realistic to assume that all particles have the same revolution frequency as this would imply either an \( \eta \)-value of 0 or a vanishing momentum spread. Thus we assume a distribution of revolution frequencies as

$$f_0 = \frac{\Delta f}{2}$$

For a subgroup of particles over a very narrow range \( df \) the total number of particles \( N \) turns to

$$d\langle I_n \rangle^2 = 2e^2 f_0^2 \frac{dN}{df} df,$$

and

$$\Rightarrow \frac{d\langle I_n \rangle^2}{df} = 2e^2 f_0^2 \frac{dN}{df}$$

(14)

The spectral density of the noise in the n-th band returns as:

$$\left( \frac{d\langle I_n \rangle^2}{df} \right)$$

in units of \([A^2/Hz]\)

integrating over the band \( f_0 \pm \Delta f \) we obtain the total noise power per band

$$\langle I_n \rangle^2 = 2e^2 f_0^2 N = 2eI_0 f_0$$

(15)

As an immediate implication we see that the total noise power in each “Schottky” band is constant. But we also know that the width of the Schottky band is increasing proportional to the harmonic number, and thus the height must decrease with \( 1/n \).

It is very clear from Fig 7 that when we continue increasing the harmonic number and thus the frequency, that at a certain point there will be Schottky band overlap.

**TRANSVERSE PHASE SPACE**

A single, highly relativistic particle in a storage ring passing through a position sensitive pick-up with infinite bandwidth generates a series a Dirac pulses. (Fig 8)

Fig. 7: A realistic picture of longitudinal Schottky bands

From what has been deduced so far for the longitudinal phase space we can already state that the following important beam parameters are measurable with Schottky noise:

- The mean revolution frequency
- The frequency distribution of particles
- The momentum spread
- The number of particles

**Fig. 8:** A single particle with betatron oscillation of amplitude \( a \) as seen by a transverse pick-up
The betatron motion results in the particle wobbling around some reference orbit and thus the transverse position changes every turn. This transverse signal is not seen in the sum output of any pick-up structure (e.g. a pair of strip-lines) but is observed in the Delta, or difference, output. The output signal of such a pick-up has two contributions, one related to the longitudinal phase space and another related to the betatron oscillations.

\[
i_{pl}(t) = \frac{e}{T} \sum_{n}^{\infty} \delta(t - nT + \phi_k) \times a_k \cos(q\omega_k t + \phi_k)
\]  

The first term under the sum sign is just the same as already seen in the longitudinal phase space discussion. In addition, however, there is an amplitude modulation of this signal due to the betatron motion. This has a frequency of \(q\omega/(2\pi)\), with \(q\) being the non-integer part of the betatron frequency and an amplitude \(a_k\) representing the oscillation amplitude. The difference response of the pick-up, \(\Delta i_{PU}\), can therefore be written as

\[
\Delta i_{PU} = S_\Delta \times a_k(t) \times i_k(t)
\]

\[
= S_\Delta \times a_k \cos(q\omega_k t + \phi_k)
\times \left[ i_0 + 2i_0 \sum_{n=1}^{\infty} \cos(n\omega_k t + n\phi_k) \right]
\]

with the term \(S_\Delta\) defining the transverse sensitivity of the pick-up.

The equation for the n-th harmonic then becomes

\[
\Delta(i_{PU})_n = S_\Delta \times a_k \times i_0 \times \left[ \cos((n-q)\omega_k t + n\phi_k) \right] + \cos((n+q)\omega_k t + n\phi_k + \phi_k)
\]

(18)

For \(N\) particles in the beam, randomly distributed both in azimuth and betatron phases, averaging equation (18) gives

\[
\left\langle \Delta i_{PU}^2 \right\rangle = S_\Delta^2 a_{rms}^2 f_0^2 \frac{N}{2} = S_\Delta^2 a_{rms}^2 e^2 f_0^2 \frac{N}{2}
\]

(19)

representing the total power (in a 1\(\Omega\) resistor) in each sideband.

![Fig. 9: Illustration of the fractional tune](image)

It can be seen from equation (19) that the total power in each band is constant and proportional to the term \(a_{rms}^2\), which for an ensemble of particles is nothing more than the rms transverse beam size and is proportional to the transverse emittance.

The width of the sidebands is given by

\[
\Delta f = (n \pm q) \times df \pm f_0 dq
\]

(20)

where \(q\) stand for the fractional tune and \(dq / Q\) is the tune spread.

![Fig. 10: Example of a Schottky signal with non-zero chromaticity](image)

The fractional part of the tune, \(q\), (Fig 9,10) can be measured using equation (20)

\[
q = \frac{1}{2} \mp \frac{f_+ - f_-}{2f_0}
\]

(21)

The tune spread \(dq / Q\) is obtained from equation (20)

\[
\frac{dq}{Q} = \frac{\Delta f_L - \Delta f_R}{2f_0 Q}
\]

(22)

With the momentum spread \(dp/p\) given by

\[
\frac{dp}{p} = \frac{1}{\eta} \times \frac{\Delta f_L + \Delta f_R}{2nf_0}
\]

(23)

Combining equations (22) and (23) it is then possible to calculate the machine chromaticity, \(\xi\):
\[ \xi = \left( \frac{dq}{Q} \right) \left( \frac{dp}{p} \right) \]  

(24)

**BUNCHEd BEAMS**

So far the discussion has been limited to the Schottky noise properties of coasting i.e. unbunched beams. For bunched beams it is necessary to convolute the transverse spectra obtained in the case of an unbunched beam with the synchrotron spectrum related to the motion of particles in the RF bucket. When particles are oscillating in an RF bucket the revolution period \( T = T_0 \) is no longer constant but modulated periodically in time with a deviation \( \Delta t \) from \( T_0 \) given as

\[ \Delta T(t) = A_s \sin(2\pi f_s t + \psi) \]  

(25)

Here \( A_s \) stands for the amplitude of the synchrotron oscillation, \( f_s \) is the synchrotron frequency and \( \psi \) is some initial phase. Introducing this time dependence into equation (1) one obtains, after some manipulations [7]:

\[ i(t) = e^{-f \psi} \sum_{n=1}^{\infty} \cos[2\pi mf_0(t) + A_s \sin(2\pi f_s t + \psi)] \]  

(26)

In other words, each single line (c.f. Fig. 5) is split up into an infinite number of modulation lines by this synchrotron oscillation related phase modulation. The spacing between adjacent lines is equal to \( f_s \). Their amplitude is given by

\[ \sum_{p=-\infty}^{\infty} J_p(2m f_0 A_s) \cos(2m f_0 + 2m f_s + p \psi) \]

with \( J_p \) being the Bessel Function of order \( p \).

In a similar way it is possible to obtain an expression [5] for the dipole moment \( D \) of a single particle travelling in an RF bucket of a circular machine

\[ D = a \cos(q2\pi f_0) e^{-f \psi} \cos[2nf_0(t) + T(t) \sin(2\pi f_s t + \psi)] \]  

(27)

It can be seen that the first cosine term in this equation is related to the amplitude modulation caused by the transverse movement, while the and the second cosine term represents the phase modulation with \( f_s \). Equation (27) can be further expanded [7] leading to a rather lengthy expression for the amplitude of each particular line.

In short one can say that for case of bunched beams each line in the transverse spectrum of the unbunched beam has to be convoluted with the synchrotron motion related spectrum. This leads to a fine structure within the distribution shown in Fig. 10. However, the total integrated power is not affected by this synchrotron motion related modulation.

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**REFERENCES**